B.Math. Hons. Second year
First semestral exam 2017
Algebra III : Instructor B.Sury

Q 1. (2+2+2+2+2 marks)
Over a commutative ring, give correct examples (without proof) of the following:
(i) a free module over a commutative ring and a submodule that is not free,
(ii) two left ideals $I \neq J$ of a noncommutative ring R which are isomorphic as left R-modules,
(iii) an integral domain which has exactly 5 maximal ideals.
(iv) a maximal ideal of $\mathbb{Z}[X, Y]$,
(v) an irreducible element of $\mathbb{R}[X, Y]/(X^2 + Y^2 - 1)$ which is not prime.

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Q 2. (8 marks)
Let R be a commutative ring with unity such that for every $a \in R$, there exists a positive integer $n(a) > 1$ so that $a^{n(a)} = a$. Prove that every prime ideal of R must be maximal.
Hint: Use characterizations of the quotient ring by prime and maximal ideals.

OR

Q 2.
Let R be a noncommutative ring with unity. Show that every two-sided ideal of the ring $M_n(R)$ must be of the form $M_n(I)$ for a two-sided ideal I of R.

OR

Q 2.
Let $\theta : \mathbb{C}[X, Y] \to \mathbb{C}[T]$ be the ring homomorphism given by $X \mapsto T^2, Y \mapsto T^3$. Prove that Ker $\theta = (X^3 - Y^2)$.

Q 3. (7 marks)
Find all the solutions to the equation $x^2 = -1$ in the division ring of real quaternions.

OR

Q 3.
Prove that the rings $\mathbb{R}[X]/(1 + X + X^2)$ and $\mathbb{C}$ are isomorphic.

OR

Q 3. Let $A = \{a/b \in \mathbb{Q} : b \text{ odd } \}$. Consider $\mathbb{Q}$ as an A-module. Show that the unique maximal ideal m of A satisfies $m\mathbb{Q} = \mathbb{Q}$. Why does this not contradict the Nakayama lemma?

Q 4. (4+4 marks)
(a) Show that if the set of non-units in a commutative ring with unity is an ideal, then the ring is a local ring.
(b) Using (a) or otherwise, prove that $\mathbb{Q}[[X]]$ is a local ring.

OR

Q 4.
Let M be an R-module where R is a commutative ring with unity. If there exists an R-module homomorphism $T : M \to M$ such that $T \circ T = T$, show that the image of T is a direct summand of M.

OR

Q 4.
Let A be a commutative ring with unity. Prove that any generating set of n elements in A^n is a basis for this free module.
Hint: Prove and use the fact that any surjective A-module homomorphism is injective.

Q 5. (8 marks)

Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1/2 & 1/3 \end{pmatrix}$. Find a matrix $P \in GL_3(\mathbb{Q})$ such that $PAP^{-1} = A^t$.

OR

Q 5.

Are the matrices $B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ similar?

Hint: You may use results on rational canonical forms.

Q 6. (9 marks)

Find a matrix which conjugates the matrix $\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$ to its Jordan form.

OR

Q 6.

Let H be the subgroup of $\mathbb{Z}^3$ generated by the elements $(2, 0, 2), (4, 6, 4)$ and $(0, 0, 18)$. Find the structure of the group $\mathbb{Z}^3/H$.

OR

Q 6.

Prove that the minimal polynomial of a matrix $A \in M_n(\mathbb{Q})$ divides its characteristic polynomial and that they have the same roots.